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Why No Analytic Continuation Connects Real-Time and Imaginary-Time Thermal Propagators

Jian Zuo[1] **and Yuan-Xing Gui**[2]

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We discuss the analytic continuation in thermal field theory. Taking advantage of the geometry of η–ξ spacetime, we explain why direct analytic continuation does not exist between the imaginary-time and real-time thermal propagators.

Finite-temperature field theory (FTFT) has two formalisms: the imaginary-time formalism and the real-time formalism (Landsman and van Weert, 1987). The imaginary-time formalism is characterized by the periodicity of imaginary time, which leads to discrete imaginary energy in the imaginary-time thermal Green functions (Matsubara, 1955; Fetter and Walecka, 1965), and the real-time formalism is charactered by the doubling of the degrees of freedom, which causes the real-time thermal Green functions to have 2×2 matrix structures (Niemi and Semenoff, 1984; Umezawa *et al.*, 1982). This paper discusses a problem concerning the connection of the two formalisms, i.e., the analytic continuation of thermal propagators. Although it has long been realized that direct analytic continuation cannot give the 2×2 matrix real-time thermal propagators from the imaginary-time thermal propagators, the reason is still not very clear. It is interesting to find that an explanation can be provided by the geometrical features of a spacetime with S^1 topology, i.e., η–ξ spacetime (Gui, 1988, 1990, 1992, 1993).

The theory of η–ξ spacetime is constructed in order to provide a unique geometrical background for FTFT. The most important parts of η–ξ spacetime are its Euclidean section and Lorentzian section. The Euclidean section has

[1] Institute of Particle Physics, Huazhong Normal University, Wuhan 430079, China; e-mail: zuoj@iopp.ccnu.edu.cn.
[2] Department of Physics, Dalian University of Technology, Dalian 116023, China; e-mail: guiyx@dlut.edu.cn.

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an S^1 topology, which makes quantum fields satisfy the periodicity for imaginary time, and the Euclidean propagators in the η–ξ spacetime correspond to the imaginary-time thermal propagators. An interesting fact about the Lorentzian section is that its geometrical structure is very similar to those of the Rindler spacetime and black hole. The infinities of the Minkowskian spacetime become "horizons" on the Lorentzian section which lead to the doubling of degrees of freedom of the fields, and the vacuum propagator in the η–ξ spacetime is equal to the 2×2 matrix real-time thermal propagator in the Minkowskian spacetime. It was suggested (Gui, 1993; Zuo and Gui, 1995) that the field theories on the Euclidean section and Lorentzian section correspond to the imaginary-time formalism and real-time formalism of FTFT, respectively. The special geometrical structures of the η–ξ spacetime also affect the relation between the two formalisms of FTFT.

This paper first gives a brief description of the structures of the η–ξ spacetime and some relations to be used. Then by discussing the procedures of solving the equations for propagators on the Euclidean section and on the Lorentzian section, respectively, we explain how the geometry influences the feasibility of analytic continuation.

The four-dimensional η–ξ spacetime can be regarded as the maximal analytic complex extension of the $S^1 \times R^3$ manifold (Gui, 1990). It has the following complex metric:

$$ds^2 = \frac{1}{\alpha^2(\xi^2 - \eta^2)}(-d\eta^2 + d\xi^2) + dy^2 + dz^2 \tag{1}$$

where $\alpha = 2\pi/\beta$ is a real constant and η, ξ, y, z are complex variables. If we limit ξ, y, z to be real and η to be a pure imaginary variable $i\sigma$, the Euclidean section of η–ξ spacetime is obtained:

$$ds^2 = \alpha^{-2}(\xi^2 + \sigma^2)^{-1}(d\sigma^2 + d\xi^2) + dy^2 + dz^2 \tag{2}$$

which under the transformation

$$\begin{aligned} \sigma &= \alpha^{-1} e^{\alpha x} \sin \alpha\tau \\ \xi &= \alpha^{-1} e^{\alpha x} \cos \alpha\tau \end{aligned} \tag{3}$$

becomes a flat Euclidean spacetime

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 \tag{4}$$

The metric (2) is singular at $\sigma = \xi = 0$, so it describes a Euclidean spacetime with $S^1 \times R^3$ topology. The periodicity of polar angle $\alpha\tau$ naturally supplies the periodicity of imaginary time in FTFT. Now let all of η, ξ, y, z be real, one gets the Lorentzian section of η–ξ spacetime. The singularities on the Lorentzian section are described by

$$\xi^2 - \eta^2 = 0 \tag{5}$$

which divides the Lorentzian section into four disjointed regions I, II, III, IV. This structure resembles that of the Schwarzschild spacetime; thus the singularities (5) are also called "horizons." Each of the regions is identified with a four-dimensional Minkowskian spacetime. One can see this from the transformation

$$\eta = \alpha^{-1} e^{\alpha x} \sinh \alpha t, \qquad \xi = \alpha^{-1} e^{\alpha x} \cosh \alpha t \tag{6}$$

which transforms region I of the Lorentzian section into

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{7}$$

Similarly, regions II, III, and IV are transformed by

$$\begin{aligned} \eta &= -\alpha^{-1} e^{\alpha x} \sinh \alpha t, & \xi &= -\alpha^{-1} e^{\alpha x} \cosh \alpha t \\ \eta &= \alpha^{-1} e^{\alpha x} \cosh \alpha t\ , & \xi &= \alpha^{-1} e^{\alpha x} \sinh \alpha t \\ \eta &= -\alpha^{-1} e^{\alpha x} \cosh \alpha t, & \xi &= -\alpha^{-1} e^{\alpha x} \sinh \alpha t \end{aligned} \tag{8}$$

respectively. The appearence of singularities (5) and the existence of several regions make it possible for the Lorentzian section to explain the doubling of degrees of freedom. While the original degrees of freedom are provided by region I, the additional degrees of freedom can be supplied by region II.

There is a relation between the transformations of regions I and II:

$$\begin{aligned} \eta &= -\alpha^{-1} e^{\alpha x} \sinh \alpha t = \alpha^{-1} e^{\alpha x} \sinh \alpha(t - i\beta/2) \\ \xi &= -\alpha^{-1} e^{\alpha x} \cosh \alpha t = \alpha^{-1} e^{\alpha x} \cosh \alpha(t - i\beta/2) \end{aligned} \tag{9}$$

Using Minkowskian coordinates (t, x, y, z), we find that the relation (9) becomes the relation between a point $a_1(t_1, x_1, y_1, z_1)$ in region I and its reflected point $a_2(t_2, x_2, y_2, z_2)$ in region II:

$$t_2 = t_1 - i\beta/2, \qquad x_2 = x_1, \qquad y_2 = y_1, \qquad z_2 = z_1 \tag{10}$$

i.e., their Minkowskian coordinates differ only in an imaginary-time interval $i\beta/2$.

Another important relation is that the direction of time t in region II is opposite to the time direction in region I. The timelike Killing fields on the Lorentzian section are defined by (Gui, 1990):

$$\left(\frac{\partial}{\partial\lambda}\right)^a = \varepsilon\alpha(\xi\eta^a + \eta\xi^a) \tag{11}$$

where $\varepsilon = 1$ and -1 for regions I and II, respectively. It is natural to choose the Killing parameter λ as the time coordinate, which coincide with Minkowskian time coordinate t in region I and $-t$ in region II.

Using the above relations, one can see how the geometry of η–ξ spacetime affects the problem of analytic continuation. The following discussion takes the example of the massless free scalar field in two-dimensional η–ξ spacetime. Simple as it is, it brings one directly into contact with the physical and geometrical essence and avoids difficult technical details. The equation for propagators of this field on the Euclidean section is

$$\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\xi^2}\right) D_{\mathrm{I}}(A - A') = -(-g_{\mathrm{E}})^{-1/2}\delta(A - A') \tag{12}$$

where g_{E} stands for the determinant of metric of the Euclidean section and D_{I} is the imaginary-time propagator. Under transformation (3), the equation becomes

$$\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial x^2}\right) D_{\mathrm{I}}\,(A - A') = -\delta(A - A') \tag{13}$$

in which the points A and A' have the coordinates (τ, x) and (τ', x'), respectively. Since the Euclidean section has S^1 topology, the propagators naturally satisfy the periodicity boundary condition:

$$D_{\mathrm{I}}(\tau - \tau') = D_{\mathrm{I}}(\tau - \tau' + \beta) \tag{14}$$

which is just the KMS condition (Kubo, 1957; Martin and Schwinger, 1959). Using this condition, we obtain the imaginary-time thermal propagator in momentum space (Fetter and Walecka, 1965):

$$D_{\mathrm{I}}(\omega_n, k) = \frac{1}{\omega_n^2 + k^2} \tag{15}$$

where $\omega_n = 2\pi n/(-i\beta)$.

Now consider the equation on the Lorentzian section:

$$\left[-\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2}\right] D_{\mathrm{R}}(A - A') = -(-g_{\mathrm{L}})^{-1/2}\delta(A - A') \tag{16}$$

where g_{L} stands for the determinant of metric of the Lorentzian section. Note that, since the regions III and IV are spacelike with respect to regions I and II, one only needs to consider the cases that A and A' are located in regions I and II. Using the Minkowskian coordinates (t, x), we can transform equation (16) into

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right] D_R(A-A') = -\delta(A-A') \tag{17}$$

Since both A and A' can be located in each of the two regions, there are four cases. If one performs a Fourier transformation of equation (17), there are four expressions for the field modes. Using the subscripts 1, 2 to stand for coordinates in regions I and II, respectively, one can write the expressions for the field modes as

$$\begin{aligned} e_{11} &= \exp\{ik(x_1 - x_1') - ik_0(t_1 - t_1')\}, \\ e_{12} &= \exp\{ik(x_1 - x_2') - ik_0(t_1 - t_2')\} \\ e_{21} &= \exp\{ik(x_2 - x_1') - ik_0(t_2 - t_1')\}, \\ e_{22} &= \exp\{ik(x_2 - x_2') - ik_0(t_2 - t_2')\} \end{aligned} \tag{18}$$

note that e_{12} and e_{21} are only formally like plane waves. But we can view the coordinates t and x in these two expressions as functions of the coordinates η and ξ; thus e_{12} and e_{21} represent the field modes on the whole Lorentzian section which relate different regions.

The Fourier coefficients $D(k, k_0)$ are different for different field modes. Hence the transformed equation takes the form of a matrix:

$$\int dk\, dk_0\, (k^2 - k_0^2)\begin{pmatrix} D_{11}e_{11} & D_{12}e_{12} \\ D_{21}e_{21} & D_{22}e_{22}\end{pmatrix} = -\int dk\, dk_0 \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22}\end{pmatrix} \tag{19}$$

By the use of coordinate relations

$$t_2 = t_1 - i\beta/2, \qquad x_2 = x_1 \tag{20}$$

one rewrites the matrix on the right side of (20) as

$$\begin{pmatrix} e^{ik(x_1-x_1')-ik_0(t_1-t_1')} & e^{ik(x_1-x_1')-ik_0(t_1-t_1'+i\beta/2)} \\ e^{ik(x_1-x_1')-ik_0(t_1-t_1'-i\beta/2)} & e^{ik(x_1-x_1')+ik_0(t_1-t_1')} \end{pmatrix} \tag{21}$$

A sign is changed in the 2–2 component because the direction of time in region I points opposite the one in region II.

The term $i\beta/2$ in the off-diagonal components is explained as a thermal factor caused by the geometrical structure of the Lorentzian section. Since regions I and II are separated on the Lorentzian section by the "horizons," they can only be connected by complex paths which run along half-circles on the Euclidean section and result in the imaginary-time interval $i\beta/2$. Physically, the observors in region I cannot measure the information in region II, which is screened by the "horizons." It is just this loss of information that makes the observers in region I find a finite temperature. In this explanation, the multiple-region geometrical structure, and thus the doubling of degrees of freedom, are the origins of the thermal factor. This geometry-induced thermal effect is quite similar to the gravity-induced Hawking–Unruh effects (Hawking, 1974; Unruh, 1975).

Using similar coordinate relations on the left side of (19), one gets four equations:

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)D_{11}(t-t',x-x')=-\delta(t-t',x-x')$$

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)D_{12}(t-t'+i\beta/2,x-x')=-\delta(t-t'+i\beta/2,x-x')$$

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)D_{21}(t-t'-i\beta/2,x-x')=-\delta(t-t'-i\beta/2,x-x')$$

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)D_{22}(t'-t,x-x')=-\delta(t'-t,x-x') \qquad (22)$$

Here the subscripts 1 for coordinates are omitted. By solving these equations, one gets

$$D_\beta(k_0,k)=\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \qquad (23)$$

where

$$D_{11}=\frac{1}{k_0^2-k^2+i\varepsilon}-\frac{2\pi i\delta(k_0^2-k^2)}{e^{\beta k_0}-1} \qquad (24)$$

$$D_{12}=\frac{2\pi i e^{-\beta k_0/2}\delta(k_0^2-k^2)}{1-e^{-\beta k_0}} \qquad (25)$$

and

$$D_{21}=D_{12}, \qquad D_{22}=-D_{11}^* \qquad (26)$$

Thus the solution of equation (16) is just the 2×2 matrix real-time thermal propagator.

According to the above derivations, one can see that direct analytic continuation between the imaginary-time and real-time thermal propagators is hindered by the different singularities on the two sections, which impose different requirements in the courses of solving these equations. On one hand, the singularity $\sigma=\xi=0$ results in the S^1 topology of the Euclidean section and thus the periodicity boundary condition (14), which requires the propagators on this section to be transformed into Fourier series. On the other hand, the Lorentzian section is divided into disjointed regions by the "horizons" (5), which cause different expressions of field modes and thus give four matrix components of the thermal propagator.

Dolan and Jackiw (1974) tried to get the real-time thermal propagators through direct analytic continuation of the imaginary-time thermal propagators. However, the result was only the 1–1 component of the 2×2 real-time thermal propagator, which leads to difficulties in calculations.

In view of the analysis in this paper, it is also clear why the early continuation was not complete. In fact, there was an implicit premise for that attempt. It was supposed that the background spacetime for the real-time formalism is the Minkowskian spacetime. This corresponds to only a region of the Lorentzian section of η–ξ spacetime, hence it cannot provide the doubling of degrees of freedom. The 1–1 component of the propagator is just the case that both A and A' are located in region I, while the other cases are ignored in that attempt. Since the singularities on the Lorentzian section, which play an important role in solving the equation for the real-time propagator, had not been considered, it is not strange that the 2×2 matrix real-time thermal propagator could not be obtained.

This paper took as an example a very simple model, and its extension to more practical calculations is far from straightforward. However, by discussing the influence of spacetime geometry on the feasibility of analytic continuation, it suggests a new intuitive way of looking at this problem.

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